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Nonunique Solutions to the Transonic Potential Flow Equation

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Introduction

IN Ref. 1 it was shown that within a certain range of angle of attack α and freestream Mach number M_∞, numerical solutions of the full-potential equation for flow past an airfoil are not unique. The emphasis in Ref. 1 was to show that the anomaly is inherent to the partial-differential equation governing the flow and not a result of its discrete representation. After extensive tests, the authors of Ref. 1 concluded that the nonuniqueness was actually a phenomenon associated with the partial-differential equation. They then conjectured that the anomaly may have a physical basis since it occurred in a range of Mach numbers where airfoils of similar thickness experience buffeting. This, they say, "raises the question of whether an instability of the outer inviscid part of the flow may also be a contributing factor" in the occurrence of buffeting. They suggested that similar investigations using the Euler or Navier-Stokes equations would shed light on the possible physical significance of the observed nonuniqueness. The purpose of this Note is twofold: First, to present results which indicate that the anomaly is due to a breakdown in the potential approximation, rather than a phenomenon associated with the inviscid flowfield; and, second, to show that the lift coefficient C_ℓ, predicted by the potential equation, is a smooth but multivalued function of the angle of attack. This second item fills a gap in the C_ℓ vs α curves presented in Ref. 1; therein, the gap was explained in terms of a hysteresis effect.

Discussion and Results

Parametric studies were conducted in the transonic range for flow past a NACA 0012 airflow using a version of the full-potential multigrid (FL036-2) code described in Ref. 2, the full-potential approximate factorization (TAIR) code described in Ref. 3, and a version of the Euler finite-volume code described in Ref. 4. Table 1 gives the typical grid size and convergence level for each of the codes used in this study. The FL036-2 code included the option of prescribing the lift coefficient and letting the angle of attack which satisfies the Kutta condition evolve as part of the solution. This option had been thoroughly tested and duplicated the results obtained by specifying the angle of attack. Reflection upon the results presented in Ref. 1 indicated that this option was needed to evaluate the C_ℓ vs α curve in the region where C_ℓ is a multivalued function of α.‡ This is the part of the curve which the authors of Ref. 1 failed to generate with α prescribed. Thus, they presented the curve with a gap between its positive- and negative-lift branches. The gap, they argued, was the result of a hysteresis effect. The part of the curve where C_ℓ is a single-valued function of α was obtained here with the TAIR code because of the poor convergence of FL036-2 in this region. The complete curve is shown in Fig. 1 for $M_\infty = 0.83$. Results from all the Euler calculations generated to date and those shown in Fig. 1 do not indicate any anomalous behavior at this Mach number; indeed, the Euler results follow closely the curve predicted by the Prandtl-Glauert thin airfoil approximation,

$$C_\ell = 2\pi\alpha/\sqrt{1-M_\infty^2}$$

The agreement between the Euler solution results and the Prandtl-Glauert theory is fortuitous, since the assumptions on which the latter is based are not valid in the supercritical regime. However, this is not the first instance where Prandtl-

Table 1 Grid size and convergence level for a typical calculation

Code	Mesh (circumf. × radial)	Residual reduced by
TAIR	149 × 30	10^{-6}
FL036-2	192 × 32	10^{-8}
Euler	121 × 35	10^{-6}

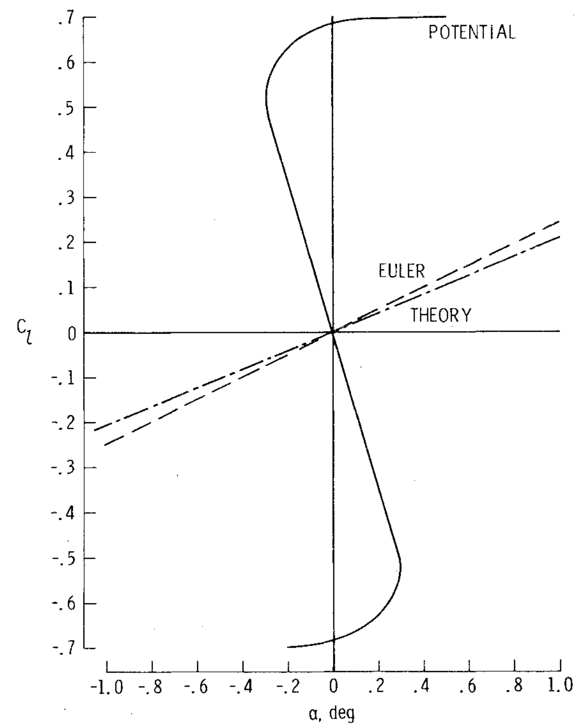

Fig. 1 Lift coefficient as a function of angle of attack computed for an NACA 0012 airfoil at $M_\infty = 0.83$ using FL036-2, TAIR, and an Euler code. The prediction of Prandtl-Glauert theory is included for comparison.

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‡It has been brought to the authors' attention that R. E. Melnik of Grumman Aerospace Corporation had independently reached the same conclusion and had similarly evaluated this part of the C_ℓ vs α curve.

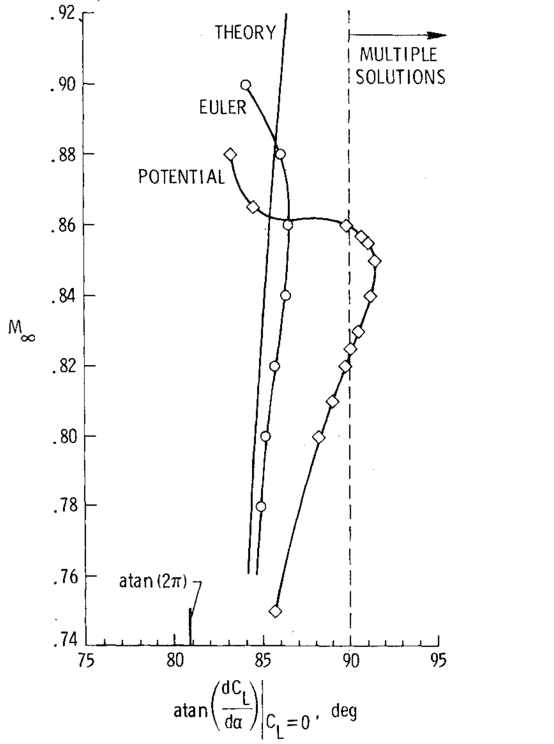

Fig. 2 Angle of the C_ℓ vs α curve at zero lift as a function of freestream Mach number computed for an NACA 0012 airfoil using FL036-2 and an Euler code. The Prandtl-Glauert theory is included for comparison.

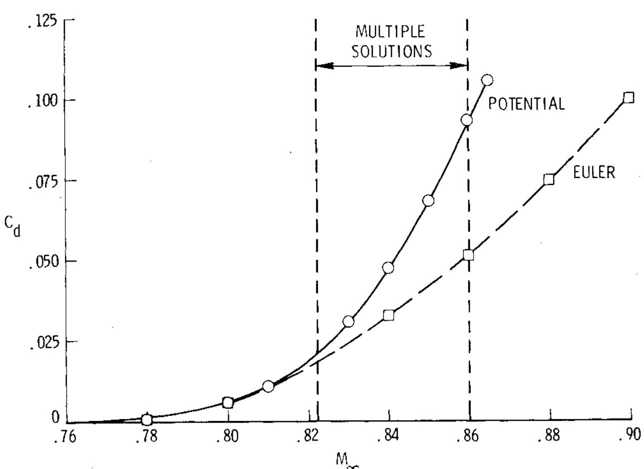

Fig. 3 Drag coefficient as a function of freestream Mach number at zero lift computed for an NACA 0012 airfoil using FL036-2 and an Euler code.

Glauert theory has provided relatively good results when applied in this region (see Ref. 5).

In order to show that the Euler solution remains well behaved throughout the transonic range, we have plotted in Fig. 2 the inclination of the C_ℓ vs α curve at zero lift as a function of M_∞. The figure clearly shows that as M_∞ is increased the Euler results remain well behaved, while the results of the potential calculations become progressively worse. For this airfoil, within the Mach number range 0.823-0.860, the slope of the potential C_ℓ vs α curve becomes negative and multiple solutions (positive, zero, and negative lift) are found for zero angle of attack. Figure 3 shows a comparison of the drag coefficient C_d as a function of M_∞ at zero angle of attack, computed with FL036-2 and the Euler code. It is worth noticing that the drag of the potential

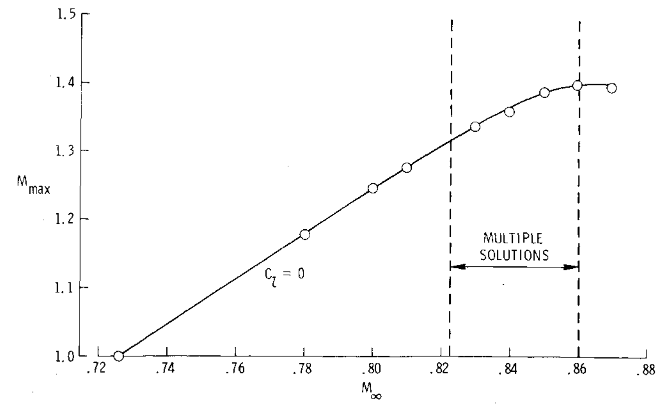

Fig. 4 Maximum local Mach number on an NACA 0012 airfoil surface as a function of freestream Mach number computed at zero lift using FL036-2.

solution becomes considerably different from that of the Euler solution for Mach numbers greater than 0.823. Figure 4 shows that for M_∞ greater than 0.82 the maximum local Mach number on the airfoil surface predicted by FL036-2 becomes greater than 1.3. This maximum local Mach number occurs just ahead of the shock wave and determines the strength of the shock. The validity of the potential approximation for rotational flows is questionable when the Mach number in front of the shock is about 1.3 or higher. The results presented in Figs. 1-4 thus seem to indicate that as the freestream Mach number increases in the supercritical range the potential approximation becomes progressively worse, eventually developing anomalous behavior in a Mach number range where rotational effects are not small and should not be neglected.

A more comprehensive study of the nonuniqueness problem is presented in Ref. 6.

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